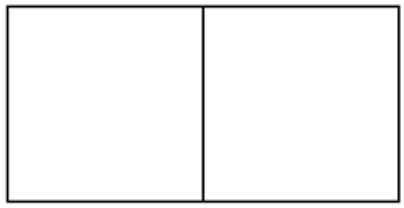
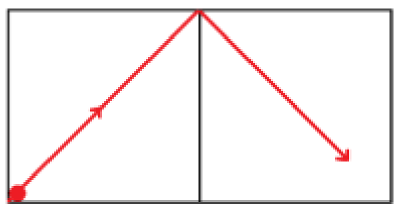
SKETCH OUTLINE: PLAYING POOL (MWW8)

The World Pool-Billiard Association has two regulation pool table sizes: the larger one has, ignoring cushions, dimensions is 254 cm by 127 cm, and the smaller one 234 cm by 117 cm. Each is twice as long as it is wide and so each is equivalent, in dimension, to two squares pasted together.



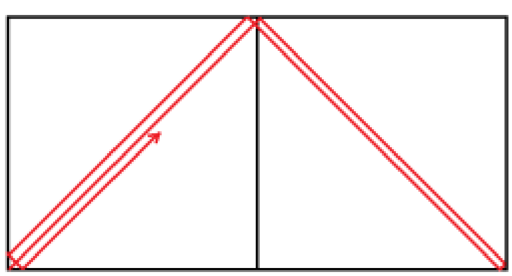
As such we can see that if we shoot a ball from the southwest corner of the table at an angle of 45 degrees it will bounce off the opposite side (and remember, angle in incidence matches the angle of reflection) and fall into the southeast pocket. Geometrically, the ball traverses the diagonals of the squares.



Regulations do allow, however, for a 3.25 millimeter variation in each of these dimensions. So here's a question.

> Suppose a large pool table is 1 mm longer and 1 mm less wide that it should be, that its dimensions are 253.9 cm by 127.1 cm. If we shoot a ball from the southwest corner at 45 degrees does it still fall into the southeast pocket?

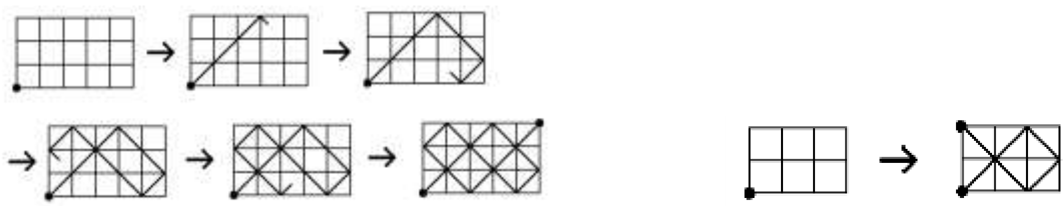
Caveat: Let's make this a theoretical question. A billiard ball has dimension and pockets have width, and so, in the real world situation, there is plenty of physical leeway and the ball will still fall into the southeast pocket. Let's ignore these physical dimensions. If we shoot a laser beam from the southwest pocket we see that it misses the southeast corner (at least at first) and will continue bouncing about the table. Will the beam fall into a corner? If so, which one?

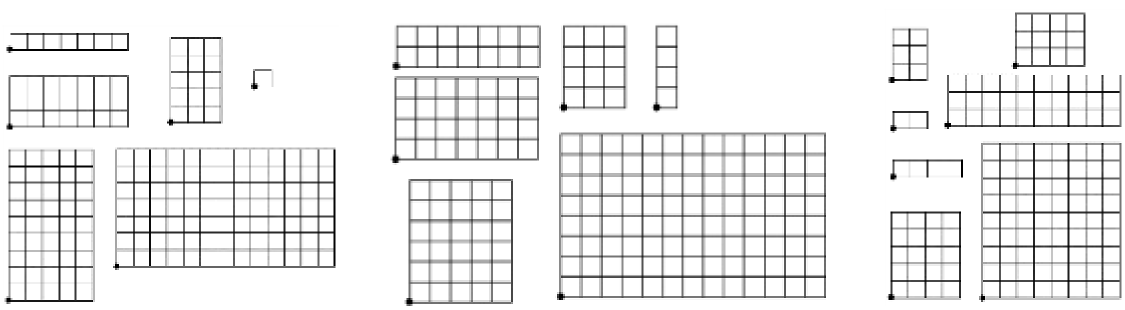


HOW TO THINK ABOUT THIS QUESTION:

A 253.9 by 127.1 can be divided into a 2539-by-1271 grid of squares, each 1 millimeter wide. So, again, we can see the path of the ball (or a beam of light) shot at 45 degrees traverses the diagonals of squares.

In MORE WITHOUT WORDS (Tarquin, 2015), James Tanton gives a wordless puzzle inviting puzzlists to develop a general theory about billiard paths. Perhaps play with these small examples first and develop a theory into which pocket a ball will fall based on some observations about its dimensions.





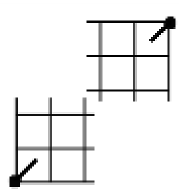
If you have a general theory in hand, perhaps you can determine into which pocket a ball will fall for each of these table dimensions too:

254.0 cm x 127.1 cm
254.1 cm x 127.1 cm
254.2 cm x 127.2 cm

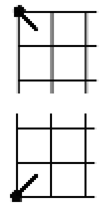
SOLUTION

We're looking at the path of a bouncing ball being shot from the southwest corner of a grid at a 45° angle.

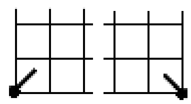
Tanton's wordless puzzle suggests that for grids with an odd count of squares in each dimension, the ball will fall into the northeast corner.



For grids an odd count of squares long and an even count wide, the ball lands in the northwest corner.

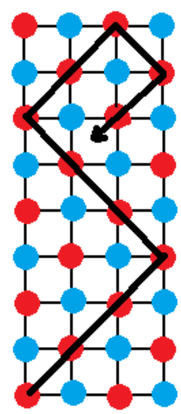


For grids an even count of square long and an odd count wide, the southwest corner.



To see why these claims hold true, color each grid point either red or blue in a checkerboard fashion, with the southwest red.

We see that the ball starting on a red point forever visits only red points.



For an odd-by-even grid as the one shown, only the northwest corner is colored red and so ball can only land in the northwest corner. For an odd-by-odd grid, one can see that only the northeast corner is colored red, and for an even-by-odd grid only the southeast corner is red. Thus the corner in which the ball lands is fixed by the evenness or oddness of the sides of the grid.

So to answer the specific pool-table questions:

253.9 cm x 127.1 cm	Northeast pocket (odd-by-odd)
254.0 cm x 127.1 cm	Southeast pocket (even-by-odd)
254.1 cm x 127.1 cm	Northeast pocket (odd-by-odd)

The 254.2 cm x 127.2 cm table is an “even-by-even” table, a case not yet discussed.

We can certainly divide this table into a 2542x1272 grid of squares, each square 1 mm wide. But since these dimensions have a common factor of 2 mm, we can also divide the table into a 1271x636 grid of squares, each square 2 mm wide. Thus this table falls into the category of an “odd-by-even” case and so the ball lands in the northwest pocket.

Further thought: Our general theory is not quite complete! How do we know that the ball does not return to start and fall back into the southwest corner? Could the ball ever enter into an infinite loop and never fall into a pocket?

In each case, the ball must retrace a diagonal of a square. Can you see how to rule out both of these possibilities simply by pondering: *If the ball retraces it's a diagonal, which was the first diagonal it retraced?*

References: Reflection paths are currently a hot topic in mathematics. See, for example, https://plus.maths.org/content/chaos-billiard-table . Because of its accessibility and surprising depth, this idea of exploring the geometry of ball motion on square grids at 45 degree angles has been a favorite topic of mathematics educators for many decades.