SKETCH OUTLINE:  DOTS ON A CIRCLE (WW10)

Surely patterns are true?

Here’s a classic puzzle. (This version of the puzzle appears in WITHOUT WORDS, Tarquin, 2015.)

[Probably better with a pizza or a pancake.]

One can guess from the psychology of the set-up here that the answer probably is not 32. What number is next in this sequence? (And then, how many pieces result if we draw seven dots on a circle and connect each and every pair of dots with a line? Eight dots? Nine?)

Warning: If you try this puzzle with a friend you might each get different answers – and not from inaccurate drawing and counting! What happens if the six dots are placed symmetrically about the circle as opposed to a haphazard arrangement? Let’s make the question of this puzzle: In placing six dots about a circle and connecting each pair of dots with a line segment, what is the maximal number of regions within the circle one could see?

Going Further: In the first five diagrams there are, respectively, 0, 1, 3, 6, and 10 lines. How many lines do you expect to see in a diagram with N dots on the circle?

In the first five diagrams there are, respectively, 0, 0, 1, and 5 intersection points. How many intersection points do you expect to see in a diagram with N dots on the circle?

In a each diagram, is there a consistent relationship between the number of lines drawn, the number of intersection points, and the number of regions one sees (assuming we are looking at the maximal number of regions possible). Can you spot that relationship? Can you explain it?
SOLUTION:

Indeed. Patterns need not be true.

If the dots are placed symmetrically about the circle, then three lines coincide and the center triangular region disappears to a point.

In this case one counts 30 regions for six dots about a circle. (And the doubling pattern 1, 2, 4, 8, 16, ... still fails!)

Using an asymmetrical placement of dots (so as to see the maximal number of pieces) one counts the following numbers of regions:

<table>
<thead>
<tr>
<th>Number of Dots</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count of Regions</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>31</td>
<td>57</td>
<td>99</td>
<td>163</td>
</tr>
</tbody>
</table>

Counting lines, intersections, and regions gives these values:

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<td>57</td>
<td>99</td>
<td>163</td>
</tr>
<tr>
<td>Number of Lines</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>28</td>
<td>36</td>
</tr>
<tr>
<td>Number of Intersections</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>15</td>
<td>35</td>
<td>70</td>
<td>126</td>
</tr>
</tbody>
</table>

Observation seems to suggest that the number of intersections, I, plus the number of lines, L, is always one less than the number of regions, R.
\[ I + L + 1 = R . \]

Another surprise lies hidden in this pattern if one is willing to be flexible in one’s play to flex the lines.

Here’s a diagram of three curved lines ( \( L = 3 \)) connecting pairs of points in a circle.

This circle is divided into 12 pieces ( \( R = 12 \)). Notice too that there 8 points of intersection within the circle ( \( I = 8 \)) and that in this picture:

\[ R = L + I + 1 \]

In fact, the same is true for most any picture you care to draw. (Try it!)

A complication lies with multiple intersections. Consider, for example:

Here \( L = 4 \) and \( R = 9 \) and it seems \( I = 2 \) invalidating our claim. But are the two intersections of “equal weight”? One of the two is a single line atop another whereas the other is three lines atop another. If we set \( I = 1 + 3 = 4 \) then our theory \( R = L + I + 1 \) again holds true!

**Conjecture:** In any diagram of curved or straight lines drawn across a circle, \( R = L + I + 1 \), as long as intersections are counted with their appropriate “weight.”

Actually, the conjecture is a theorem!

**Proof:** First notice that the equation \( R = L + I + 1 \) certainly holds for a diagram with no lines ( \( R = 1 \), \( L = 0 \), and \( I = 0 \)). Now let’s see if it remains valid as we draw new lines, adding one line at a time. In starting any new line note that two things occur as soon it intersects itself or intersects a pre-existing line: the count of intersections increases by one and the count of regions increases by one. The formula \( R = L + I + 1 \) remains balanced. And this equation remains valid when the new line returns to the circumference of the circle: the count of lines increases by one which is balanced by the fact that a final region is split in two. The formula \( R = L + I + 1 \) holds all the way through the process of drawing any new line.
So this does it! The equation \[ R = L + I + 1 \] starts out valid and never changes as lines are added. It remains true then for all diagrams.

**GETTING A FORMULA:**

Back to the number of regions created by straight line segments connecting each and every pair of dots placed on a circle.

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We certainly have \[ R = L + I + 1 \], so the challenge lies in finding formulas for \( L \) and \( I \). First of all, notice that in this straight line case, each line determines and is determined by a choice of pair of boundary dots:

![Diagram](image)

The formula for the number of ways to select two objects from \( n \) is:

\[
L = \frac{n!}{2!(n-2)!} = \frac{1}{2}n(n-1)
\]

Each intersection point determines and is determined by a choice of four boundary points. (Think about this.)

![Diagram](image)

The formula for selecting four objects from \( n \) is:

\[
I = \frac{n!}{4!(n-4)!} = \frac{1}{24}n(n-1)(n-2)(n-3)
\]

Thus the sequence of “region numbers” is given by:

\[
R = L + I + 1 = 1 + \frac{1}{2}n(n-1) + \frac{1}{24}n(n-1)(n-2)(n-3).
\]

A friendly looking formula!