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CURIOUS MATHEMATICS FOR FUN AND JOY



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THIS MONTH'S PUZZLER:

Consider the numbers $\{0, 1, 2, 3, 4\}$ and their rearrangement $\{4, 3, 2, 1, 0\}$. Adding these two lists together termwise gives $\{4, 4, 4, 4, 4\}$, nothing but the square number four.

$$\{0, 1, 2, 3, 4\} \quad \{4, 3, 2, 1, 0\} \quad \{4, 4, 4, 4, 4\}$$

Adding $\{0, 1, 2, 3, 4, 5\}$ and its rearrangement $\{0, 3, 2, 1, 5, 4\}$ termwise, gives $\{0, 4, 4, 4, 9, 9\}$, nothing but square numbers.

$$\{0, 1, 2, 3, 4, 5\} \quad \{0, 3, 2, 1, 5, 4\} \quad \{0, 4, 4, 4, 9, 9\}$$

For each counting number N is there sure to be some rearrangement $\{0, 1, 2, \dots, N\}$ so that adding the two lists together termwise gives nothing but square numbers?

▲▲▲▲▲ SQUARE PERMUTATIONS

In 1978, in his paper "Square permutations" (*Mathematics Magazine*, Vol 51, No 1, January 1978, pp 64-66) Benjamin Schwartz introduced the idea of the opening puzzle. (He said the idea is originally due to David Silverman.) Do square permutations always exist?



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If we omit zero from our counting numbers, the answer is certainly no. For example,

No permutation of $\{1\}$ adds to give squares.

No permutation of $\{1, 2\}$ adds to give squares.

No permutation of $\{1, 2, 3, 4\}$ adds to give squares.

No permutation of $\{1, 2, 3, 4, 5, 6\}$ adds to give squares.

No permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ adds to give squares.

No permutation of $\{1, 2, 3, \dots, 11\}$ adds to give squares.

(To see why no permutation exists for the final example, for instance, consider which elements we need to add to 4 and to 11 to make perfect squares.)

One can check that appropriate permutations of $\{1, 2, 3, \dots, N\}$ exists for all other values up to 11.

$$\{1, 2, 3\} + \{3, 2, 1\}$$

$$\{1, 2, 3, 4, 5\} + \{3, 2, 1, 5, 4\}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8\} + \{8, 7, 6, 5, 4, 3, 2, 1\}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\} + \{8, 2, 6, 5, 4, 3, 2, 9, 1, 7\}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} + \{3, 2, 1, 5, 4, 10, 9, 8, 7, 6\}$$

How about for values greater than 11? Can we construct a permutation that works for $N = 12$?

Here's one possible strategy. Think of 16, the first square number larger than 12.

Adding 4 to 12 gives 16 and we can use this to generate a partial list of termwise sums that work.

$$\begin{array}{r} 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \\ + \ 12 \ 11 \ 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \\ \hline 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \end{array}$$

We're missing the terms 1, 2, and 3, but we know a permutation that works just for those three! Combining, we get a solution for $N = 12$.

$$\begin{array}{r|l} 1 \ 2 \ 3 & 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \\ + \ 3 \ 2 \ 1 & 12 \ 11 \ 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \\ \hline 4 \ 4 \ 4 & 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \end{array}$$

This technique fails, alas, for $N = 13$. Adding 3 to 13 gives us the next square number, suggesting we work with a permutation of $\{3, 4, \dots, 13\}$.

$$\begin{array}{r} 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \\ + \ 13 \ 12 \ 11 \ 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \\ \hline 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \end{array}$$

But we cannot juxtapose this with a working permutation of $\{1, 2\}$.

Nonetheless, some trial-and-error shows that $N = 13$ does have a valid square permutation.

$$\{1, 2, 3, \dots, 13\} + \{8, 2, 13, 12, 11, 10, 9, 1, 7, 6, 5, 4, 3\}$$

One can check by hand, by using the technique described above (which sometimes works) or by trial-and-error, that there are valid permutations for all N between 12 and 20.

Can we go higher?

Well, let's understand when the technique described works and when it fails. Here again is the picture that worked for $N = 12$.

$$\begin{array}{r|l} 1 \ 2 \ 3 & 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \\ + \ 3 \ 2 \ 1 & 12 \ 11 \ 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \\ \hline 4 \ 4 \ 4 & 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \ 16 \end{array}$$

We are looking for a permutation of $\{1, 2, 3, \dots, N\}$ that gives only square values upon termwise addition. Let k^2 be the smallest square number not smaller than N . (So $N \leq k^2$.)

$$\begin{array}{cccc|cccc}
 1 & 2 & 3 & & k^2 - N - 1 & & k^2 - N & \dots & N \\
 + & * & * & * & * & & N & \dots & k^2 - N \\
 \hline
 & & & \dots & & & k^2 & \dots & k^2
 \end{array}$$

If we know there is a permutation that works for $\{1, 2, \dots, k^2 - N - 1\}$, then we'll have a permutation that works for $\{1, 2, \dots, N\}$.

What are the restrictions on k ?

We need $k^2 - N$ to be a number no more than N , so we must have $k^2 - N \leq N$, that is, $k^2 \leq 2N$.

We also need $k^2 - N - 1$ to be among the values we know do have a permutation. We know that these exist for a whole string of values from 12 onwards. So let's see if we can arrange for $k^2 - N - 1$ to be ≥ 12 , that is, for $k^2 \geq N + 13$.

Alright, for each value of $N \geq 12$, is it true that there is a square number k^2 with $N + 13 \leq k^2 \leq 2N$?

No! At least not for $N = 12, 13, 14, 15, 16$, or 17 . Bother!

But might it hold for $N = 18$ and above? (This would be fine, as we know there are permutations that work from $N = 12$ to $N = 20$.)

So here is our refined question:

If $N \geq 18$ and k^2 is the smallest square number satisfying $N + 13 \leq k^2$, does it then follow that $k^2 \leq 2N$?

Well, let's see.

Suppose k is the smallest integer satisfying $k^2 \geq N + 13$, that is, satisfying $k \geq \sqrt{N + 13}$. Then $k = \lceil \sqrt{N + 13} \rceil$, and so k is no larger than $\sqrt{N + 13} + 1$. Thus

$$\begin{aligned}
 k^2 &\leq (\sqrt{N + 13} + 1)^2 \\
 &= N + 13 + 2\sqrt{N + 13} + 1 \\
 &= N + 14 + 2\sqrt{N + 13}.
 \end{aligned}$$

We are hoping this is smaller than $2N$. So now we are wondering: *Is $14 + 2\sqrt{N + 13}$ less than N ?* That is, we are wondering if

$$2\sqrt{N + 13} \leq N - 14,$$

that is, if

$$4(N + 13) \leq N^2 - 28N + 196,$$

that is, if

$$0 \leq N^2 - 32N + 144,$$

that is, if

$$0 \leq (N - 16)^2 - 112.$$

This is the case if we have $N - 16 \geq \sqrt{112}$, which we do if $N \geq 28$.

Ooh! That's a different range than $N \geq 18$!

TAKING STOCK

I am starting to get lost in details, in particular, the fine details that don't quite fit together. What do we have so far?

* We know that there are valid square permutations of $\{1, 2, \dots, N\}$ for $N = 12$ up to $N = 20$. (They can be checked by hand.)

* We know that if we can find a square number k^2 that fits this table,

$$\begin{array}{cccc|cccc}
 1 & 2 & 3 & & k^2 - N - 1 & & k^2 - N & \dots & N \\
 + & * & * & * & * & & N & \dots & k^2 - N \\
 \hline
 & & & \dots & & & k^2 & \dots & k^2
 \end{array}$$

then we have a square permutation of $\{1, 2, \dots, N\}$.

* By “fits this table” we mean that $k^2 - N \leq N$ and that a valid square permutation for $\{1, 2, \dots, k^2 - N - 1\}$ is already known to exist.

* We know that if $N \geq 28$, then there is square number k^2 with $k^2 - N \leq N$ and with $k^2 - N - 1$ at least 12.

This feels like enough to prove

Theorem: For each $N \geq 12$ there is a valid square permutation of $\{1, 2, \dots, N\}$.

Proof: One can check by hand that square permutations exist for all values $N = 12$ up to $N = 28$.

Here’s how to construct a square permutation for $N = 29$.

Choose the smallest square number $k^2 \geq N + 13$. Then combine the known square permutation for $\{1, 2, \dots, k^2 - N - 1\}$ (and here $k^2 - N - 1$ is a value somewhere between 12 and $N - 1$) with the permutation $\{N, \dots, k^2 - N\}$ of $\{k^2 - N, \dots, N\}$. One then has a square permutation of $\{1, 2, \dots, N\}$.

$$\begin{array}{rcccc|cccc}
 1 & 2 & 3 & & k^2 - N - 1 & & k^2 - N & \dots & N \\
 + & * & * & * & * & & N & \dots & k^2 - N \\
 \hline
 & & & \dots & & & k^2 & \dots & k^2
 \end{array}$$

Repeat this construction to get a square permutation of $N = 30$, then of $N = 31$, and so on, indefinitely.

CHALLENGE: Solve the opening puzzler. Prove that square permutations of $\{0, 1, 2, \dots, N\}$ exist for all values of N . (This time there are no exceptional cases.)

CHALLENGE: Prove that for each $N \geq 1$ there is a permutation of $\{1, 2, \dots, N\}$ so that each termwise sum is a prime number.

Ben Schwartz proves this first challenge in his paper and discusses ideas for proving the second.


RESEARCH CORNER

Explore other types of permutations of $\{1, 2, \dots, N\}$ or of $\{0, 1, 2, \dots, N\}$. Are there ones so that each and every termwise sum is a cube or just some odd power? Each is a triangular number? Each is a congruent to 0 or 1 modulo 8? And so on.


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